SOLUTIONS OF PROBLEMS.

2689 [April, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

Show that the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}$$

is $y_1 = (\cos \theta - \rho)/(\cos \theta + \rho)$ where $\rho = \sqrt{\sin^2 \varphi - \sin^2 \theta}$.

This problem was suggested to the proposer by a professor of civil engineering, and has important applications in the theory of conjugate stresses.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}. \tag{1}$$

Taking the logarithm of (1), differentiating and setting the result equal to zero we get

$$\cot x - \tan (x + \varphi) + \tan (x + \theta) - \cot (x + \varphi + \theta) = 0 \tag{2}$$

or

$$\cot x - \cot (x + \varphi + \theta) = - \tan (x + \theta) + \tan (x + \varphi);$$

whence

$$\frac{\sin (\varphi + \theta)}{\sin (\varphi - \theta)} = \frac{\sin x \sin (x + \varphi + \theta)}{\cos (x + \varphi) \cos (x + \theta)}. \tag{3}$$

Or writing (2) in the form

$$\cot x + \tan (x + \theta) = \tan (x + \varphi) + \cot (x + \varphi + \theta)$$

we get

$$\frac{\cos \theta}{\sin x \cos (x + \theta)} = \frac{\cos \theta}{\cos (x + \varphi) \sin (x + \varphi + \theta)};$$

or

$$\frac{\cos (x + \varphi)}{\cos (x + \theta)} = \frac{\sin x}{\sin (x + \varphi + \theta)}. \tag{4}$$

(4) in (3) gives

$$\frac{\cos (x + \varphi)}{\sin x} = \sqrt{\frac{\sin (\varphi - \theta)}{\sin (\varphi + \theta)}};$$

whence

$$\cot x = \tan \varphi + \sec \varphi \sqrt{\frac{\sin (\varphi - \theta)}{\sin (\varphi + \theta)}}. \tag{5}$$

By (4) and (1), we may write

$$y_1 = \frac{\sin^2 x}{\sin^2 (x + \varphi + \theta)}$$

so that by (5)

$$\frac{1}{\sqrt{y_1}} = \cos (\varphi + \theta) + \tan \varphi \sin (\varphi + \theta) + \sec \varphi \sqrt{\sin (\varphi + \theta) \sin (\varphi - \theta)} = \frac{\cos \theta + \rho}{\cos \varphi}.$$

$$\therefore \quad y_1 = \frac{\cos^2 \varphi}{(\cos \theta + \rho)^2} = \frac{\cos^2 \theta - \rho^2}{(\cos \theta + \rho)^2} \equiv \frac{\cos \theta - \rho}{\cos \theta + \rho}.$$

To prove that we have a maximum we find:

$$\frac{dy}{dx} = \frac{\cos \theta}{\cos (x + \theta) \sin (x + \varphi + \theta)} \left\{ \frac{\cos (x + \varphi)}{\cos (x + \theta)} - \frac{\sin x}{\sin (x + \varphi + \theta)} \right\}$$

$$= \frac{\cos \theta}{\cos^2 (x + \theta) \sin^2 (x + \varphi + \theta)} \{\sin \varphi \cos (2x + \varphi + \theta) + \sin \theta\},$$

which shows that we have either a maximum or a minimum when the brace equals zero, since it is the only variable factor of odd power. Choosing the least value of x that will make the brace zero we have that $\cos (2x + \varphi + \theta)$ decreases as x increases so that the sign of dy/dx changes from positive to negative as x passes through this critical value.